How do we represent hyperplanes and spheres resp. spheres in the Euclidean resp. Spherical model?

Euclidean model

The hyperplane $\langle n,x\rangle = d$ with $n \in S^{n-1}$ and $d \in \mathbb{R}$ becomes $\hat{p} = n + 0\cdot e_0 - 2d\, e_\infty$ and

$$(n,x) = d \iff \langle \hat{p}, e_0 + x + \|x\|^2 e_\infty\rangle = 0 \text{ with}$$

$$\langle \hat{p},\hat{p}\rangle = (n,n) > 0 .$$

A hypersphere $(x-m, x-m) = r^2$ with $m \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$ becomes $\hat{s} = e_0 + m + (\|m\|^2 - r^2)e_\infty$, since

$$(x-m, x-m) = r^2 \iff \langle \hat{s}, e_0 + x + \|x\|^2 e_\infty\rangle = 0$$

$$\Updownarrow$$

$$\cdot(-2) \quad \langle e_0 + m + (\|m\|^2 - r^2)e_\infty, e_0 + x + \|x\|^2 e_\infty\rangle = 0$$

$$\iff (x,m) - \frac{1}{2}(\|m\|^2 - r^2 + \|x\|^2) = 0$$

<u>Note:</u> Hyperplanes as well as hyperspheres are sections of $Q_0^n \subseteq \mathbb{R}^{n+1,1}$ with hyperplanes. The corresponding sections of $Q_1^n \subset \mathbb{R}^{n+1,1}$ with the same hyperplanes yield the images of the hyperplanes/-spheres after stereographic projection. → 111a

<u>Proposition</u> The angle of intersection of two spheres $\hat{s}_1$ and $\hat{s}_2$ with $\hat{s}_1 = e_0 + m_1 + (\|m_1\|^2 - r^2)e_\infty$ and $\hat{s}_2 = e_0 + m_2 + (\|m_2\|^2 - r^2)e_\infty$ is given by

$$\cos\alpha = \frac{\langle \hat{s}_1, \hat{s}_2\rangle}{\sqrt{\langle \hat{s}_1,\hat{s}_1\rangle\langle \hat{s}_2,\hat{s}_2\rangle}}$$

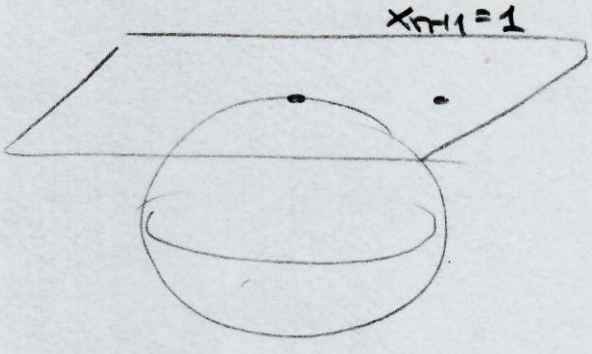
Stereographic projection maps planes to hyperspheres on S^n through the northpole:

$$\text{hyperplane in } \mathbb{R}^n \rightsquigarrow [n + 2d e_\infty] \in P(\mathbb{R}^{n+1,1})$$

$$= [n + d e_{n+1} + d e_{n+2}]$$

$$= \Big[\underbrace{\tfrac{n}{d} + e_{n+1}}_{} + e_{n+2}\Big]$$

poles point in tangent plane to north pole.



If $d = 0$ then $[n + 0 \cdot e_\infty]$ is a great circle through the north-pole.

Stereographic projection map hyperspheres to hyperspheres on S^n. This is also a renormalization of the corresponding poles in $P(\mathbb{R}^{n+1,1})$

$$\text{hypersphere in } \mathbb{R}^n \rightsquigarrow [e_0 + m + (\|m\|^2 - r^2) e_\infty]$$

$$= \left[\tfrac{1}{2}(e_{n+2} - e_{n+1}) + m + \tfrac{1}{2}(\|m\|^2 - r^2)(e_{n+2} + e_{n+1})\right]$$

$$= \left[m + \left(-\tfrac{1}{2} + \|m\|^2 - r^2\right) e_{n+1} + \left(\tfrac{1}{2} + \|m\|^2 - r^2\right) e_{n+2}\right]$$

$$= \begin{cases} \text{great circle} & \text{if } \tfrac{1}{2} + \|m\|^2 - r^2 = 0 \\ \dfrac{1}{\left(\frac{1}{2} + \|m\|^2 - r^2\right)} \begin{pmatrix} m \\ -\frac{1}{2} + \|m\|^2 - r^2 \end{pmatrix} & \text{otherwise.} \end{cases}$$

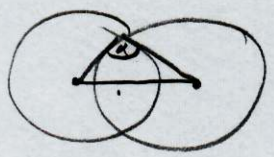
Proof.

$$\langle \hat{S}_1, \hat{S}_2 \rangle = (m_1, m_2) - \frac{1}{2}\left(\|m_1\|^2 - r_1^2 + \|m_2\|^2 - r_2^2 \right)$$

$$= -\frac{1}{2}\left((m_1 - m_2, m_1 - m_2)^2 - (r_1^2 + r_2^2) \right)$$

$$= -\frac{1}{2}(-2 r_1 r_2 \cos\alpha)$$

$$= r_1 r_2 \cos\alpha$$



$$\|m_1 - m_2\|^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\alpha$$

$$\langle \hat{S}_1, \hat{S}_1 \rangle = (m_1, m_1) - (\|m_1\|^2 - r^2)$$

$$= r^2$$

$$\Rightarrow \frac{\langle \hat{S}_1, \hat{S}_2 \rangle}{\sqrt{\langle \hat{S}_1, \hat{S}_1 \rangle \langle \hat{S}_2, \hat{S}_2 \rangle}} = \cos\alpha .$$

RD.

Corollary: Two spheres intersect orthogonally iff.
$\langle \hat{S}_1, \hat{S}_2 \rangle = 0$.

Möbius transformations in the projective model

The projective transformations of $P(\mathbb{R}^{n+1,1})$ that map the quadric to itself are in the group $PO(n+1,1)$. They map hyperplanes to hyperplanes and thus hyperspheres to hyperspheres in the resp. models. So

$$PO(n+1,1) \leq Mob(n).$$

We need to show that all Möbius transformations are in $PO(n+1,1)$.

<u>Theorem</u>: The group of Möbius transformations Möb(n) is isomorphic to PO(n+1, 1).

<u>Proof</u>. We have already seen that $PO(n+1,1) \leq \text{Möb}(n)$. Now we prove the other inclusion by showing that the generators of Möb(n) in the Euclidean model belong to PO(n+1, 1). In fact they are proj. reflections preserving the quadric, i.e. of the form

$$x \xmapsto{R_p} x - 2\frac{\langle p, x\rangle}{\langle p, p\rangle} p \text{ for } \langle p, p\rangle > 0.$$

These maps preserve the quadric, since:

$$\langle R_p x, R_p x\rangle = \langle x - 2\frac{\langle p, x\rangle}{\langle p, p\rangle} p, \ldots\rangle$$

$$= \underbrace{\langle x, x\rangle}_{=0} - 4\frac{\langle p, x\rangle^2}{\langle p, p\rangle} + 4\frac{\langle p, x\rangle^2}{\langle p, p\rangle} = 0.$$

Consider the above reflection for $p = e_0 + m + (\|m\|^2 - r^2)e_\infty = \hat{s}$ and a point $x \in \mathbb{R}^n \rightsquigarrow \hat{x} = e_0 + x + \|x\|^2 e_\infty$:

$$\hat{x} \mapsto \hat{x} - 2\frac{\langle \hat{x}, \hat{s}\rangle}{\langle \hat{s}, \hat{s}\rangle}\hat{s}$$

- $\langle \hat{s}, \hat{s}\rangle = \|m\|^2 - (\|m\|^2 - r^2) = r^2 > 0$
- $\langle \hat{s}, \hat{x}\rangle = (m, x) - \frac{1}{2}(\|x\|^2 + \|m\|^2 - r^2)$
$= -\frac{1}{2}(\|m - x\|^2 - r^2).$

$$= e_0 + x + \|x\|^2 e_\infty + \frac{\|m-x\|^2 - r^2}{r^2}(e_0 + m + (\|m\|^2 - r^2)e_\infty)$$

$$= \frac{\|m-x\|^2}{r^2} e_0 + (x - m) + \frac{\|m-x\|^2}{r^2} m + \alpha e_\infty$$

$$\sim \underbrace{e_0 + \frac{r^2}{\|m-x\|^2}(x-m) + m}_{\text{sphere inversion}} + \tilde{\alpha} e_\infty.$$

Proof (cont.)

In case of the reflection in a hyperplane in $\mathbb{R}^n$ the calculation is the following:

$$p = n + 2de_\infty, \quad \hat{x} = e_0 + x + \|x\|^2 e_\infty.$$

$$\hat{x} \longmapsto \hat{x} - 2\frac{\langle \hat{x}, n+2de_\infty\rangle}{\langle n+2de_\infty, n+2de_\infty\rangle}(n+2de_\infty)$$

$$= e_0 + x + \|x\|^2 e_\infty - 2\frac{(x,n)-d}{(n,n)}(n+2de_\infty)$$

$$= e_0 + \underbrace{x - 2\frac{(x,n)-d}{(n,n)}n}_{\text{reflection in an affine hyperplane.}} + \beta e_\infty \ldots$$

$$\Rightarrow \text{Möb}(n) \leq PO(n+1,1). \qquad \boxtimes$$

Corollary The Möbius transformations on the sphere are generated by inversion in spheres that intersect the unit sphere orthogonally.

Proof. 1. $\langle \hat{s}, \hat{s}_0\rangle = 0 \Leftrightarrow \langle e_0 + m + (\|m\|^2 - r^2)e_\infty, e_0 - e_\infty\rangle = 0$

$\Leftrightarrow \|m\|^2 - r^2 = 1. \quad \|m\|^2 - 1 = r^2$

$$y \in S^n \leadsto y + e_{n+2} \longmapsto y + e_{n+2} - 2\frac{\langle \hat{s}, y+e_{n+2}\rangle}{\langle \hat{s}, \hat{s}\rangle}\hat{s}$$

$$= y + e_{n+2} - 2\frac{(m,y)-1}{r^2}(m+e_{n+2})$$

$$= y - m + \left(\frac{(m-y, m-y)}{r^2}\right)(m+e_{n+2})$$

$$\sim \underbrace{m + \frac{r^2}{(m-y, m-y)}(y-m)}_{\text{inversion}} + e_{n+2}$$

$$\boxtimes$$