We have established a close relationship of Möbius transformations and projective transformations by identifying $\mathbb{R}^2 \cup \{\infty\} \cong \mathbb{CP}^1$.

The orientation preserving Möbius transformations are exactly the projective transformation in $PGL(2,\mathbb{C})$.
To obtain all Möbius transformation we need to consider complex conjugation as well.
This correspondence has the following consequences.

(i) orientation preserving Möbius transformations preserve the cpx. cross-ratio, i.e.

$$cr(z_1,z_2,z_3,z_4) = cr(f(z_1), f(z_2), f(z_3), f(z_4)).$$

If orientation is reversed we have

$$cr(z_1,z_2,z_3,z_4) = \overline{cr(f(z_1), f(z_2), f(z_3), f(z_4))}.$$

(ii) For arbitrary three pairs of points $z_i \to w_i$ $(i=1,2,3)$ there exists an or. pres. Möbius transf. s.t. $f(z_i)=w_i$, because $d+2=3$ pts define a proj. transf. uniquely. Similarly for orientation reversing transf.

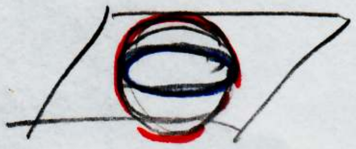
(iii) An arbitrary circle can be mapped to the real axis. Since cross-ratio is preserved this implies that

$$cr(z_1,z_2,z_3,z_4) \in \mathbb{R} \iff z_1,z_2,z_3,z_4 \text{ lie on a circle}.$$

Spherical model:

Unify hyperspheres and hyperplanes by a stereographic projection $\mathbb{R}^n \cup \{\infty\} \to S^n$.

- hyperspheres S^{n-1} in $\mathbb{R}^n$ $\longrightarrow$ hyperspheres S^n in $\mathbb{R}^{n+1}$



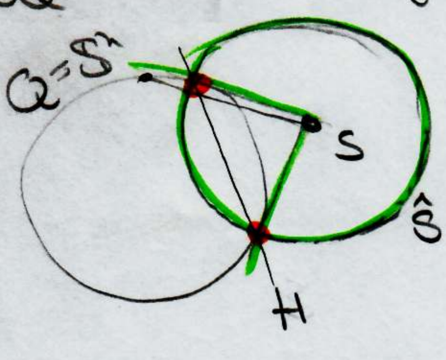
$\longrightarrow$ hyperspheres S^n intersecting S^n orthogonally

$\longrightarrow$ hyperspheres $S^{n-1} \subseteq S^n$.

- hyperplanes in $\mathbb{R}^n$ $\longrightarrow$ hyperspheres through the north pole.

$\longrightarrow$ orthogonal intersection of S^n with another n-sphere.

Identification of hyperspheres in S^n with points outside

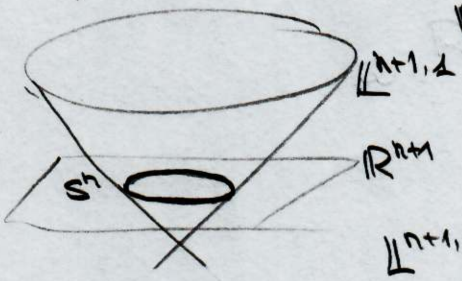


- A hypersphere in S^n is the intersection of $S^n \cap H$.
- H polar hyperplane to S wrt. S^n.
- $S^n \cap \hat{S} = S^n \cap H = \{ y \in \mathbb{R}^{n+1} \mid y \in S^n, (y, s) = 1 \}$

$\to$ Möbius transformations of S^n are generated by inversions in the hyperspheres $\hat{S}$ intersecting S^n orthogonally.

Projective model for Möbius geometry

The unit sphere is the affine image of the quadric $Q \subseteq P(\mathbb{R}^{n+1,1})$:



$y \in S^n \subseteq \mathbb{R}^{n+1}$

$\Leftrightarrow \left\langle \begin{pmatrix} y \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right\rangle = 0.$

$\Leftrightarrow \left[\begin{pmatrix} y \\ 1 \end{pmatrix}\right] \in Q$

$\mathbb{L}^{n+1,1} = \{x \in \mathbb{R}^{n+1,1} \mid \langle x, x \rangle = 0\}$

Using polarity wrt. the quadric Q we identify points outside the quadric, i.e. $\langle s, s\rangle_{n+1,1} > 0$ a point outside Q then the polar plane $H_s = \{[x] \in P(\mathbb{R}^{n+1,1}) \mid \langle x, s \rangle = 0\}$ defines a hypersphere in S^n.

Looking at the hyperplane $x_{n+2} = 1$ we recover the spherical model for Möbius geometry. $S^n = Q_1^n = \mathbb{L}^{n+1,1} \cap \{x_{n+2}=1\}$

Where is the Euclidean model?

Introduce new coordinates:

$$e_0 = \frac{1}{2}(e_{n+2} - e_{n+1}) \text{ and } e_\infty = \frac{1}{2}(e_{n+2} + e_{n+1}).$$

Then $\{e_1, \dots, e_n, e_0, e_\infty\}$ is a basis of $\mathbb{R}^{n+1,1}$ with

$$\langle e_i, e_j \rangle = \delta_{ij} \text{ for } i,j \in \{1, \dots, n\}$$

$$\langle e_i, e_0 \rangle = \langle e_i, e_\infty \rangle = 0 \quad \forall i \in \{1, \dots, n\}$$

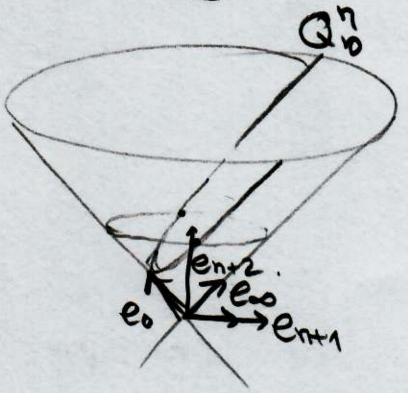
$$\langle e_0, e_0 \rangle = \langle e_\infty, e_\infty \rangle = 0 \text{ and}$$

$$\langle e_0, e_\infty \rangle = \frac{1}{4}(\langle e_{n+2}, e_{n+2} \rangle - \langle e_{n+1}, e_{n+1} \rangle) = \frac{1}{4}(-2) = -\frac{1}{2}.$$

The intersection of light-cone with the hyperplane $e_0 + x + x_\infty e_\infty$, $x \in \mathbb{R}^n$, $x_\infty \in \mathbb{R}$ or $\langle v, e_\infty \rangle = -\frac{1}{2}$ is a paraboloid, since

$$\langle e_0 + x + x_\infty e_\infty, e_0 + x + x_\infty e_\infty \rangle = 0$$

$$\Longleftrightarrow \|x\|^2 - 2\tfrac{1}{2} x_\infty = 0 \Longleftrightarrow x_\infty = \|x\|^2 .$$



So we have a bijective map from $\mathbb{R}^n$ to the paraboloid

$$Q_0^n = L^{n+1,1} \cap \{ v \in \mathbb{R}^{n+1,1} \mid \langle v, e_\infty \rangle = -\tfrac{1}{2} \} .$$

(The point ∞ is mapped to e_∞)

Claim: the identification of $S^n = Q_1^n$ and $\mathbb{R}^n \cong Q_0^n$ along the generators of the light-cone is the stereographic projection.

$e_{n+1} = e_\infty - e_0$
$e_{n+2} = e_\infty + e_0$

Proof. $[e_0 + x + \|x\|^2 e_\infty] = [y + e_{n+2}] = [y' + (1 - y_{n+1}) e_0 + (1 + y_{n+1}) e_\infty]$

$$= \left[\frac{y'}{1 - y_{n+1}} + e_0 + \frac{1 + y_{n+1}}{1 - y_{n+1}} e_\infty \right]$$

$$\blacktriangleright \quad \frac{1 + y_{n+1}}{1 - y_{n+1}} = \frac{1 - y_{n+1}^2}{(1 - y_{n+1})^2} = \frac{\|y'\|^2}{(1 - y_{n+1})^2} .$$

So $x = \dfrac{y'}{1 - y_{n+1}}$ is exactly the stereographic projection.

What happens to the north pole, i.e. $\begin{pmatrix} y' \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\Rightarrow [0 + e_{n+1} + e_{n+2}] = [e_\infty] . \quad ☺$$