Klein's Erlangen program:

Geometry should be considered

→ the study of invariants under a grp of transf. ←

Different geometries and their invariants

- Euclidean, $\mathbb{R}^n$, $x \mapsto Ax+b$, $A\in O(n)$, $b\in\mathbb{R}^n$; distances & angles.

$\uparrow$

- Affine geom., $\mathbb{R}^n$, $x \mapsto Ax+b$, $A\in GL(n)$, $b\in\mathbb{R}^n$; ratio's of distances, parallelity

$\uparrow$

- Projective, $\mathbb{RP}^n$, $PGL(n+1)$; cross-ratio's, planes, quadrics, incidences
- hyperbolic, $\mathbb{H}^n_{pr}$, $PO(n,1)$, hyperbolic distances and angles.

Hierarchy of transformations:

$$A \in PGL(n+1) \longrightarrow \begin{pmatrix} \tilde{A} & b \\ 0 & 1 \end{pmatrix} \text{ affine transf. } \tilde{A}\in GL(n)$$

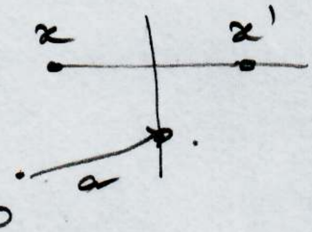
$$\longrightarrow \begin{pmatrix} \tilde{A} & b \\ 0 & 1 \end{pmatrix} \text{ Euclidean transf } \tilde{A}\in O(n).$$

Möbius geometry

Consider $\mathbb{R}^n$ with the standard Euclidean scalar product: $(x,y) = \sum x_i y_i$.

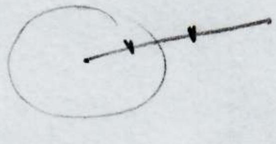
Reflection in a hyperplane $\{x \in \mathbb{R}^n \mid (x-a, n) = 0\}$

$$x \longmapsto x - 2\frac{(x-a,n)}{(n,n)}n.$$



Inversion in a sphere $\{x \in \mathbb{R}^n \mid (x-m, x-m) = r^2\}$

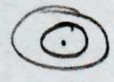
$$x \longmapsto m + \frac{r^2}{(x-m,x-m)}(x-m)$$



Definition A Möbius-transformation of $\mathbb{R}^n \cup \{\infty\}$ is a composition of reflections in hyperplanes and inversions in hyperspheres.
They build a group called Möbius group Möb(n).

- Inversions become reflections when $r \longrightarrow \infty$.
- hyperplanes can be considered spheres with infinite radius.

Examples of Möbius-transformations:

1. translations 
2. rotations 
3. Scaling .

<u>Remark</u>: Euclidean transformations and Similarity transformations are subgrps of Möb(n).

<u>Theorem</u>: Möbius transformations are conformal and map hyperspheres and hyperplanes to hyperspheres and hyperplanes.

<u>Questions</u>:

- Which group is Möb(n)?
- What is the relation to projective geometry?

Is Möbius geometry a subgeometry of proj. geometry?

<u>Theorem</u>: Any bijective map $f: \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ that maps hyperspheres/planes to hyperspheres/planes is a Möbius transformation.

<u>Proof</u> <u>Case 1</u>: $f(\infty) = \infty$.

$\Rightarrow$ f maps hyperplanes to hyperplanes

$\Rightarrow$ f maps lines to lines

So f is an affine map that maps hyperspheres to hyperspheres.

$\Rightarrow$ f is a similarity transformation and hence a Möbius transf.

<u>Case 2</u> $f(\infty) = q \in \mathbb{R}^n$. Let $g: \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$ be an inversion in a sphere with center q. Then $g \circ f$ is a Möbius transformation by case 1.

$\Rightarrow f = g \circ (g \circ f)$ is a Möbius transformation. $\boxtimes$

Two-dimensional Möbius geometry

We can identify $\mathbb{R}\cup\{\infty\} \cong \mathbb{C}\cup\{\infty\} = \mathbb{CP}^1$ with the complex projective line.

Similarity transformations can be written as

$$z \longmapsto az+b \quad \text{or} \quad a\bar{z}+b.$$

Sphere inversion is of the form:

$$z \longmapsto z_0 + \frac{r^2}{|z-z_0|^2}(z-z_0) = \frac{z_0\bar{z}_0 - |z_0|^2 + r^2}{\bar{z}-\bar{z}_0}$$

Reflections are:

$$z \longmapsto z - 2\frac{\mathrm{Re}(z-p)\cdot\bar{n}}{|n|^2}\,n$$

$$= \cancel{z} - \frac{(\cancel{z}-p)\bar{n} + \overline{(z-p)}\,n}{\bar{n}}$$

$$= p - n\bar{z} + \bar{p}n/\bar{n} \ldots$$

$\langle z-p, n\rangle = (z-p)_1\cdot n_1 + (z-p)_2\cdot n_2$

$\mathrm{Re}\, w = \frac{1}{2}(w+\bar{w})$.

So the Möbius transformations are proj transformations of $\mathbb{CP}^1$ compose with the map $z \mapsto \bar{z}$.

Proposition The Möbius transformations of $\hat{\mathbb{C}} = \mathbb{CP}^1$ are precisely the maps of the form:

$$z \longmapsto \frac{az+b}{cz+d} \quad \text{or} \quad \frac{a\bar{z}+b}{c\bar{z}+d} \quad \text{with } a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0.$$

Proof. We have already shown, that all Möbius transf. are of the above form.

On the other hand:

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

which is the composition of Möbius transformations

$$z \xrightarrow{\text{similarity}} cz+d \xrightarrow{1/z} \frac{1}{cz+d} \xrightarrow{\text{similarity}} \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

Similarly for $\bar{z}$.

$\boxtimes$